LAYER POTENTIALS METHODS

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Contents

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1. Introduction

In this lecture, we introduce the layer potentials methods to solve Dirichlet and Neumann boundary problems of Laplace equation.

The note organizes as follow. In section2,we introduce our problems and some notations. Section3 deals with the general kernel properties on the boundry. Section4 and 5 study the double and single layer potentials respectively. Then we solve our problems in section2 at the functional analysis frame in section6. Finally, we select some common facts about removable singularity and asymptotic behavior at infinity of harmonic function and its radical derivative in appendix.

2. Setup

Let Ω always be a bounded open subset in $\mathbb{R}^n$ with C^2 boundary S,and we set $\Omega' = \mathbb{R}^n \setminus \Omega$. Ω and Ω' will both be allowed to be disconnected, however S can be differetiable there can only be finitely many component. We denote the components of Ω by $\Omega_1, \cdots, \Omega_m$ and those of Ω' by $\Omega'_0, \Omega'_1, \cdots, \Omega'_{m'}$, where Ω'_0 is the unbounded component.

By tubular neiborhood lemma, we have following C^1 natural diffeomorphism

$$F(x,t) = x + t\gamma(x) \tag{2.1}$$

from $S \times (-\epsilon, \epsilon)$ to a neiborhood V of S for some $\epsilon > 0$ and every point in $\{x + t\gamma(x) : t \in (-\epsilon, \epsilon)\}$ is the unique nearest point to x. Here, $\gamma(x)$ is outward-pointing unit normal vector field along S.

The we can extend the normal derivative to the whole tubular neiborhood V. we set

$$\partial_\gamma u(x + t\gamma(x) = \gamma(x) \cdot \nabla u(x + t\gamma(x)) \tag{2.2}$$

for $u \in C^1(V)$.

We define $C_\gamma(\Omega)$ to be the functions $u \in C^1(\Omega) \cap C(\bar{\Omega})$ such that the limit

$$\partial_{\gamma-} u(x) = \lim_{t<0, t\to 0} \gamma(x) \cdot \nabla u(x + t\gamma(x)) \tag{2.3}$$

exists for each $x \in S$,the convergence being uniform on S.

Similarly, we define $C_\gamma(\Omega')$ to be the functions $u \in C^1(\Omega') \cap C(\bar{\Omega}')$ such that the limit

$$\partial_{\gamma+} u(x) = \lim_{t>0, t\to 0} \gamma(x) \cdot \nabla u(x + t\gamma(x)) \tag{2.4}$$

exists for each $x \in S$,the convergence being uniform on S. The operators $\partial_{\gamma-}$ and $\partial_{\gamma+}$ are called the interior and exterior normal derivatives on S.

If $\Omega \subset \mathbb{R}^n \setminus \{0\}$, we set

$$\tilde{\Omega} = \{|x|^{-2}x : x \in \Omega\}$$

and if u is a function on Ω, we define its **Kelvin transform** $\tilde{u}$, a function on $\tilde{\Omega}$ by

$$\tilde{u} = |x|^{2-n}u(|x|^{-2}x)$$

Suppose u is harmonic outside some bounded set, we say u is harmonic at infinity if $\tilde{u}$ has a removable singularity at 0.

We can now state our problems we propose to solve:

The Interior Dirichlet Problem: Given $f \in C(S)$,find $u \in C(\bar{\Omega})$ such that u is harmonic on Ω and $u = f$ on S.

The Exterior Dirichlet Problem: Given $f \in C(S)$,find $u \in C(\bar{\Omega}')$ such that u is harmonic on $\Omega \cup \{\infty\}$ and $u = f$ on S.

The Interior Neumann Problem: Given $f \in C(S)$,find $u \in C_\gamma(\Omega)$ such that u is harmonic on Ω and $\partial_{\gamma-}u = f$ on S.

The Exterior Neumann Problem: Given $f \in C(S)$,find $u \in C_\gamma(\Omega')$ such that u is harmonic on $\Omega \cup \{\infty\}$ and $\partial_{\gamma+}u = f$ on S.

To begin with, we prove the uniqueness for four problems.

Theorem 2.1. *(I) If u solves the interior Dirichlet problem with $f = 0$,then $u = 0$.*
(II) If u solves the exterior Dirichlet problem with $f = 0$, then $u = 0$.
(III) If u solves the interior Neumann problem with $f = 0$, then u is constant on each component of Ω.
(IV) If u solves the exterior Neumann problem with $f = 0$, then u is constant on each component of Ω, and $u = 0$ unbounded component Ω_0' when $n > 2$.

Proof. (I)(II) is trivial by maximum principle.(III) just by Green's formula. We just need to prove (IV). Let $r > 0$ such that $\bar{\Omega} \subset B_r$, by Green's formula

$$\int_{B_r \setminus \Omega} |\nabla u|^2 = -\int_{B_r \setminus \Omega} u\Delta u - \int_S u\partial_{\gamma+}u + \int_{\partial B_r} u\partial_r u$$

Since $|u(x)| = O(|x|^{2-n})$ and $|\partial_r u(x)| = O(|x|^{1-n})$ for $n > 2$, $|u(x)| = o(\log(|x|))$ and $|\partial_r u(x)| = O(|x|^{-2})$ for $n = 2$. Then let $r \to \infty$, yields our desired. □

We shall see the Dirichlet problem are always solvable. For the Neumann problems, however, there are some necessary conditions as follows.

Theorem 2.2. *(I)If the interior Neumann problem has a solution, then $\int_{\partial\Omega_j} f = 0$ for $j = 1, \cdots, m$.*
(II)If the exterior Neumann problem has a solution, then $\int_{\partial\Omega_j'} f = 0$ for $j = 1, \cdots, m'$, and also for $j = 0$ in case $n = 2$.

Proof. For $n = 2$,Let $r > 0$ such that $\Omega \subset B_r$, by Green's formula

$$\int_{\partial B_r} \partial_r u - \int_{\partial\Omega_0'} \partial_{\gamma+}u = 0$$

As before, let $r \to \infty$, yields our desired. The remainder is trivial by Green's formula. $\square$

3. Kenel Properties on Boundary

Let K be a measurable function on $S \times S$ and suppose $0 < \alpha < n-1$. We call K a kenel of order α if

$$K(x,y) = A(x,y)|x-y|^{-\alpha} \tag{3.1}$$

where $A(x,y)$ is a bounded function on $S \times S$. We call K a kenel of order zero if

$$K(x,y) = A(x,y)\log|x-y| + B(x,y) \tag{3.2}$$

where A and B is a bounded function on $S \times S$.

We call K a continuous kenel of order α $(0 \leq \alpha < n-1)$ if K is a kenel of order α and K is continuous on $\{(x,y) \in S : x \neq y\}$.

If K is a kenel of order α $(0 \leq \alpha < n-1)$, we define the operator T_K formally by

$$T_K f(x) = \int_S K(x,y) f(y) d\sigma(y) \tag{3.3}$$

We select some useful properties as following lemma.

Lemma 3.1. *If K is a kenel of order α $(0 \leq \alpha < n-1)$*
(I)T_K is bounded on $L^p(S)$ for $1 \leq p \leq \infty$.
(II)If K is supported in $\{(x,y) : |x-y| < \epsilon\}$, there is a constant $C = C(n,\alpha,S)$ such that

$$\begin{aligned}
\| T_K f \|_p &\leq C\epsilon^{n-1-\alpha} \| A \|_\infty \| f \|_p \quad (\alpha > 0) \\
\| T_K f \|_p &\leq C\epsilon^{n-1}(\| A \|_\infty (1 + |\log \epsilon|) + \| B \|_\infty) \| f \|_p \ (\alpha = 0)
\end{aligned}$$

(III)If $K \in C(S \times S)$,T_K is a compact map from $L^p(S)$ into $C(S)$.
(IV)T_K is compact on $L^2(S)$. Moreover, if K is continuous kernel, T_K is compact on $L^p(S)$ for $1 \leq p \leq \infty$.
(V)If K a continuous kenel of order α $(0 \leq \alpha < n-1)$, then T_K transforms bounded function into continuous function.
(VI)If K a continuous kenel of order α $(0 \leq \alpha < n-1)$, then T_K transforms L^p into continuous function for $p > (n-1)/(n-1-\alpha)$.
(VII)If K a continuous kenel of order α $(0 \leq \alpha < n-1)$, if $u \in L^2(S)$ and $u + T_K u \in C(S)$, then $u \in C(S)$.

Proof. (II) implies (I) by choose $\epsilon > diam(S)$.
(II):WLOG, $\epsilon \leq j(S) \leq 1$, here $j(S)$ is the injective radius of S. Using polar coordinates on S centered at x, we see $(\alpha > 0)$:

$$\int_S |K(x,y)| d\sigma(y) \quad \leq \quad \| A \|_\infty \int_{|x-y|<\epsilon} |x-y|^\alpha dy$$

$$
\begin{aligned}
&\leq C(n,S) \parallel A \parallel_\infty \int_0^\epsilon r^{n-2-\alpha} dr \\
&= C(n,S,\alpha) \parallel A \parallel_\infty \epsilon^{n-1-\alpha}
\end{aligned}
$$

Similarly,

$$\int_S |K(x,y)| d\sigma(x) \leq C(n,S,\alpha) \parallel A \parallel_\infty \epsilon^{n-1-\alpha}$$

in the case $\alpha = 0$:

$$
\begin{aligned}
\int_S |K(x,y)| d\sigma(y) &\leq \parallel A \parallel_\infty \int_{|x-y|<\epsilon} |\log(|x-y|)| dy + \parallel B \parallel_\infty \epsilon^{n-1} \\
&\leq C(n,S) \parallel A \parallel_\infty \int_0^\epsilon -r^{n-2} \log r dr + \parallel B \parallel_\infty \epsilon^{n-1} \\
&= C(n,S)(\parallel A \parallel_\infty |\log \epsilon| + 1 + \parallel B \parallel_\infty) \epsilon^{n-1}
\end{aligned}
$$

the remainder is obvious by Young's inequality.
(III):By computation

$$|\int_S K(x,z) f(z) dz| \leq C(K,S) \parallel f \parallel_p$$

$$|\int_S (K(x,z) - K(y,z)) f(z) dz| \leq C(S) |K(x,z) - K(y,z)| \parallel f \parallel_p$$

then Ascoli-Arzela theorem yields our desired.
(IV):Given $\epsilon > 0$, set $K_\epsilon(x,y) = K(x,y)$ if $|x-y| < \epsilon$ and $K_\epsilon = 0$ otherwise, and set $K'_\epsilon(x,y) = K(x,y) - K'_\epsilon(x,y)$. It's easy to see $T_{K'_\epsilon}$ is compact operator on $L^2(S)$(if K is continuous kernel, $T_{K'_\epsilon}$ is compact on $L^p(S)$ for $1 \leq p \leq \infty$). Then conclusion holds by closeness of compact operator.
(V):Given $x \in S$ and $\delta > 0$. Define $B_\delta(x) = \{y \in S : |x-y| < \delta\}$, we have
$\alpha > 0$:

$$
\begin{aligned}
|T_K f(x) - T_K f(y)| &= |\int_S (K(x,z) - K(y,z)) f(z) dz| \\
&\leq \int_{B_\delta(x)} |(K(x,z) - K(y,z)) f(z) dz| + \int_{S \setminus B_\delta(x)} |(K(x,z) - K(y,z)) f(z)| \\
&\leq \parallel A \parallel_\infty \parallel f \parallel_\infty \left(\int_{B_{2\delta}(x)} |x-z|^\alpha + \int_{B_{2\delta}(y)} |y-z|^\alpha \right) + R(x,y,\delta) \\
&= C(n,\alpha) \delta^{n-1-\alpha} \parallel A \parallel_\infty \parallel f \parallel_\infty + R(x,y,\delta)
\end{aligned}
$$

$\alpha = 0$:

$$
\begin{aligned}
|T_K f(x) - T_K f(y)| &= |\int_S (K(x,z) - K(y,z)) f(z) dz| \\
&\leq \int_{B_\delta(x)} |(K(x,z) - K(y,z)) f(z) dz| + \int_{S \setminus B_\delta(x)} |(K(x,z) - K(y,z)) f(z)|
\end{aligned}
$$

$$\begin{aligned} &\leq \ \| A \|_\infty \| f \|_\infty \left(\int_{B_{2\delta}(x)} |x-z|^\alpha + \int_{B_{2\delta}(y)} |y-z|^\alpha \right) + R(x,y,\delta) \\ &= \ C(n,\alpha) \| f \|_\infty \delta^{n-1} (\| A \|_\infty |\log \delta| + 1 + \| B \|_\infty) + R(x,y,\delta) \end{aligned}$$

then the conclusion holds by $\epsilon - \delta$ argument.
(VI)As same as (V).
(VII)Given $\epsilon > 0$, choose $\phi \in C(S \times S)$ such that $0 \leq \phi \leq 1$, $\phi(x,y) = 1$ for $|x-y| < \frac{1}{2}\epsilon$ and $\phi(x,y) = 0$ for $|x-y| < \epsilon$. Set $K_0 = K\phi$ and $K_1 = K - K_0$, then

$$|T_{K_1}u(x) - T_{K_1}u(y)| \leq \| u \|_2 \left[\int_S (K_1(x,z) - K_1(y,z)) f(z) dz \right]^{\frac{1}{2}}$$

then $f := u + T_{K_0}u$ is continuous. If we choose ϵ sufficiently small, then u is continuous by (II). □

4. Double Layer Potentials

In this section, we introduce the double layer potential and study its kernel behavior partially by some facts in section3.

Let $\phi \in C(S)$, the double layer potential with moment ϕ given by

$$u(x) = \int_S \partial_{\gamma_y} \Gamma(x,y) \phi(y) d\sigma(y) \tag{4.1}$$

for $x \in \mathbb{R}^n \setminus S$, here $\Gamma(x)$ is the fundamental solution.
We note that

$$\partial_{\gamma_y} \Gamma(x,y) = -\frac{(x-y)\cdot\gamma(y)}{\omega_n |x-y|^n}$$

We point out that u is not in $\mathbb{R}^n$ and there is a jump at boundry. Before proving the important fact, we need some basic lemma.

Lemma 4.1. *(I)There is a constant $c > 0$, such that for all $x, y \in S$*

$$|(x-y)\cdot\gamma(y)| \leq c|x-y|^2$$

(II)K is a continuous kernel of order $n-2$ on S.
(III)The following integal formula hold

$$\int_S \partial_{\gamma_y} \Gamma(x,y) d\sigma(y) = \begin{cases} 1 & if \ x \in \Omega, \\ 0 & if \ x \in \Omega'. \end{cases}$$

$$\int_S K(x,y) d\sigma(y) = \frac{1}{2} \qquad if \quad x \in S$$

(IV)There is a constant $C > 0$ such that for all $x \in \mathbb{R}^n \setminus S$

$$\int_S |\partial_{\gamma_y} \Gamma(x,y)| d\sigma(y) \leq C$$

(V)Suppose $\phi \in C(S)$ and $\phi(x_0) = 0$ for some $x_0 \in S$, then u continuous at x_0.

Proof. (I):WLOG $y=(0,\cdots,0)$ and $\gamma_y=(0,\cdots,1)$, $x_n=f(x_1,\cdots,x_{n-1})$, then by Taylor's theorem

$$|(x-y)\cdot\gamma_y|\leq c|(x_1,\cdots,x_{n-1})|^2\leq c|x-y|^2$$

S is compact and of class C^2, so a uniform c exists.
(II):By definition of double layer potential and (I).
(III):The first integal formula is trivial by property of fundamental solution. Now we prove the second, suppose $x\in S$, we set

$$S_\epsilon=S\setminus(S\cap B_\epsilon),\quad \partial B'_\epsilon=\partial B_\epsilon\cap\Omega,\quad \partial B''_\epsilon=\{y\in\partial B_\epsilon:\gamma_x\cdot y<0\}$$

Green's formula gives

$$0=\int_{S_\epsilon}K(x,y)d\sigma(y)+\int_{\partial B'_\epsilon}\partial_{\gamma_y}\Gamma(x,y)d\sigma(y)$$

here we have chosen proper oritation on $\partial B'_\epsilon$. Then

$$\int_S K(x,y)d\sigma(y)=-\lim_{\epsilon\to0}\int_{\partial B'_\epsilon}\partial_{\gamma_y}\Gamma(x,y)d\sigma(y)=\lim_{\epsilon\to0}\frac{\epsilon^{n-1}}{\omega_n}\int_{\partial B'_\epsilon}d\sigma(y)$$

Again since S is C^2, the symmetry difference between $\partial B'_\epsilon$ and $\partial B''_\epsilon$ is contained in

$$\{y\in\partial B_\epsilon:|y\cdot\gamma_x|\leq c(\epsilon)\},\quad c(\epsilon)=O(\epsilon^2)$$

whose area is $O(\epsilon^n)$ and the result follows.
(IV):Let $dist(x,S)$ be the distance from x to the nearest point in S. Fix δ with the following property:
a.$\delta<\frac{1}{2c}$, here c is in (I).
b.$\delta<\frac{1}{2}\epsilon$, here ϵ is in section2 about tubular neiborhood.
Case1:$dist(x,S)\geq\frac{1}{2}\delta$. Then $|\partial_{\gamma_y}\Gamma(x,y)|\leq C(n)\delta^{1-n}$, hence

$$\int_S|\partial_{\gamma_y}\Gamma(x,y)|d\sigma(y)\leq C(n,S)$$

Case2:$dist(x,S)\leq\frac{1}{2}\delta$. Let x_0 be the nearest point to x in S.If $y\in S\setminus B_\delta(x_0)$:

$$|x-y|\geq|y-x_0|-|x-x_0|\geq\frac{1}{2}\delta$$

then the integal in $S\setminus B_\delta(x_0)$ is bounded by $C(n,S)$ as above.If $y\in B_\delta(x_0)$, we note that:

$$\begin{aligned}\omega_n|\partial_\gamma\Gamma(x,y)| &= \frac{|(x-y)\cdot\gamma_y|}{|x-y|^n}\\ &\leq \frac{|(x-x_0)|+c|y-x_0|^2}{|x-y|^n}\end{aligned}$$

and

$$\begin{aligned}|x-y|^2 &= |x-x_0|^2+|y-x_0|^2+2(x-x_0)(y-x_0)\\ &\geq |x-x_0|^2+|y-x_0|^2-2c|(x-x_0)||(y-x_0)|\end{aligned}$$

$$\geq \frac{1}{2}(|x-x_0|^2+|y-x_0|^2)$$

hence

$$|\partial_\gamma \Gamma(x,y)| \leq C(n)\left[\frac{|x-x_0|}{(|x-x_0|^2+|y-x_0|^2)^{n/2}}+\frac{c}{|y-x_0|^{n-2}}\right]$$

therefore, we set $a=|x-x_0|$ and integrate in polar coordinate:

$$\begin{aligned}\int_{B_\delta(x_0)} |\partial_{\gamma_y}\Gamma(x,y)|d\sigma(y) &\leq C(n,S)\int_0^\delta \left[\frac{a}{(a^2+r^2)^{n/2}}+\frac{1}{r^{n-2}}\right] r^{n-2}dr \\ &= C(n,S)\left[\int_0^\infty \frac{r^{n-2}}{(1+r^2)^{n/2}}dr+\delta\right] \\ &= C(n,S)\end{aligned}$$

Combining this with integal in $B_\delta(x_0)$, so we have done.
(V):Given $\epsilon>0$, by

$$\begin{aligned}|u(x)-u(x_0)| \leq \int_{B_\delta(x_0)} (|\partial_{\gamma_y}\Gamma(x,y)|+|\partial_{\gamma_y}\Gamma(x_0,y)|)|\phi(y)|d\sigma(y) \\ +\int_{S\setminus B_\delta(x_0)} (|\partial_{\gamma_y}\Gamma(x,y)|-|\partial_{\gamma_y}\Gamma(x_0,y)|)|\phi(y)|d\sigma(y)\end{aligned}$$

we choose δ sufficiently small to make the first item less than $\frac{1}{2}\epsilon$, then choose $\eta>0$ and $|x-x_0|<\eta$, so the sum less than ϵ.

□

Suppose $\phi \in C(S)$ and u is defined by (4.1). we define the function u_t on S for small $t \neq 0$ by

$$u_t(x)=u(x+t\gamma(x))$$

Then we have following jump formula.

Theorem 4.2. *Suppose $\phi \in C(S)$ and u is defined by (4.1). The restriction of u to Ω has a continuous extension to $\bar{\Omega}$ and the restriction u of Ω' has a extension to $\bar{\Omega}'$.More presicely, the function u_t converge uniformly on S to continuous u_- and u_+ as t approach zero from below and above,respectively. u_- and u_+ are given by*

$$u_- = \frac{1}{2}\phi(x)+\int_S K(x,y)\phi(y)d\sigma(y) \tag{4.2}$$

$$u_+ = -\frac{1}{2}\phi(x)+\int_S K(x,y)\phi(y)d\sigma(y) \tag{4.3}$$

i.e

$$u_-=\frac{1}{2}\phi+T_K\phi; \quad u_+=-\frac{1}{2}\phi+T_K\phi \tag{4.4}$$

Moreover, $\phi=u_--u_+$.

Proof. If $x \in S$ and $t < 0$ is sufficiently small, then $x + t\gamma_x \in \Omega$, so

$$\begin{aligned} u_t(x) &= \phi(x) \int_S (\partial_{\gamma_y} \Gamma(x + t\gamma_x, y) d\sigma(y) \\ &+ \int_S (\partial_{\gamma_y} \Gamma(x + t\gamma_x, y)(\phi(y) - \phi(x)) d\sigma(y) \\ &= \phi(x) + \int_S (\partial_{\gamma_y} \Gamma(x + t\gamma_x, y)(\phi(y) - \phi(x)) d\sigma(y) \end{aligned}$$

the second integal is continuous in t as $t \to 0$. Hence

$$\begin{aligned} \lim_{t<0, t\to 0} u_t(x) &= \phi(x) + \int_S K(x,y)\phi(y) d\sigma(y) - \phi(x) \int_S K(x,y) d\sigma(y) \\ &= \frac{1}{2}\phi(x) + \int_S K(x,y)\phi(y) d\sigma(y) \end{aligned}$$

If $t > 0$, the argument is same except that

$$\phi(x) \int_S (\partial_{\gamma_y} \Gamma(x + t\gamma_x, y) d\sigma(y) = 0$$

The convergence is uniform by the proof of (V) in lemma4.1. □

5. Single Layer Potentials

We now consider the single layer potential with moment ϕ

$$u(x) = \int_S \Gamma(x,y)\phi(x) d\sigma(y) \tag{5.1}$$

here $\phi \in C(S)$. It's easy to see that the restriction of $\Gamma(x,y)$ to $S \times S$ is a cntinuous kernel of order $n-2$, so u is also well defined on S. Now we consider the normal derivative of u, for $x \in V \setminus S$ we have

$$\partial_\gamma u(x) = \int_S \partial_{\gamma_x} \Gamma(x,y)\phi(y) d\sigma(y) \tag{5.2}$$

We set

$$K^*(x,y) = K(y,x) \tag{5.3}$$

and

$$T_K^* f(x) = \int_S K^*(x,y) f(y) d\sigma(y) \tag{5.4}$$

then T_K^* is the adjoint of T_K as an operator on $L^2(S)$.

First, we prove some simple facts which used in Theoerm 5.2 and section6.

Lemma 5.1. *If $\phi \in C(S)$ and u defined by (5.1), then*
(a1)$u \in C(\mathbb{R}^n)$.
(a2)If $\frac{1}{2}\phi + T_K^*\phi = f$, then $\int_S \phi = \int_S f$.
Moreover $n = 2$, we have
(b1)u is harmonic at infinity iff $\int_S \phi = 0$, in which case u vanishes at infinity.
(b2)If $\int_S \phi = 0$, and u is constant on Ω, then $\phi = 0$.

Proof. a1:We need to show continuity on S, $x_0 \in S$ then

$$\begin{aligned} |u(x) - u(x_0)| \leq & \int_{B_\delta(x_0)} (|\Gamma(x,y)| + |\Gamma(x_0,y)|)|\phi(y)|d\sigma(y) \\ & + \int_{S \setminus B_\delta(x_0)} |\Gamma(x,y) - \Gamma(x_0,y)||\phi(y)|d\sigma(y) \end{aligned}$$

the remaining argument is same as proof of lemma4.1.
a2:By

$$\begin{aligned} \int_S f(x)d\sigma(x) &= \frac{1}{2}\int_S \phi(x) + \int_S \int_S K(y,x)\phi(y)d\sigma(y)d\sigma(x) \\ &= \int_S \phi(x) \end{aligned}$$

the last equality due to Fubini's theorem and (III) of lemma4.1.
b1:We have

$$u(x) = \frac{1}{2\pi}\int_S (\log|x-y| - \log|x|)\phi(y)d\sigma(y) + \frac{1}{2\pi}\log|x|\int_S \phi(y)d\sigma(y)$$

the first item tends to 0 uniformly for $y \in S$ as $x \to \infty$, then conclusion is obvious.
b2:If $u = c$ on $\bar{\Omega}$, u solves the exterior Dirichlet problem with $f = c$ by b1. Thus $u = c$ everywhere, so $\phi = 0$ by the following theoerm.

□

As might be expected, there is a jump discontinuity between the quantities defined by (5.2) on $V \setminus S$ and by (5.4) on S. Indeed, we have following theoerm.

Theorem 5.2. *Suppose $\phi \in C(S)$ and u defined on $\mathbb{R}^n$. The restriction of u to $\bar{\Omega}$(resp.$\bar{\Omega}'$) is in $C_\gamma(\Omega)$(resp.$C_\gamma(\Omega')$), and for $x \in S$ we have*

$$\partial_{\gamma-}u(x) = -\frac{1}{2}\phi(x) + \int_S K(y,x)\phi(y)d\sigma(y) \tag{5.5}$$

$$\partial_{\gamma+}u(x) = \frac{1}{2}\phi(x) + \int_S K(y,x)\phi(y)d\sigma(y) \tag{5.6}$$

i.e

$$\partial_{\gamma-}u = -\frac{1}{2}\phi + T_K^*\phi; \quad \partial_{\gamma+}u = \frac{1}{2}\phi + T_K^*\phi \tag{5.7}$$

Moreover, $\phi = \partial_{\gamma+}u(x) - \partial_{\gamma-}u(x)$.

Proof. Let v is the double layer potential with moment ϕ, consider the following f on the tubular neiborhood V of S by

$$f(x) = \begin{cases} v(x) + \partial_\gamma u(x) & if\ x \in V \setminus S, \\ T_K\phi(x) + T_K^*\phi(x) & if\ x \in S. \end{cases}$$

Claim:f is continuous on V.
proof of claim: it suffices to show that if $x_0 \in S$ and $x = x_0 + t\gamma_{x_0}$, then $f(x) - f(x_0) \to 0$ as $t \to 0$, the convergence being uniform in x_0. By

$$\begin{aligned} |f(x) - f(x_0)| &= |\int_S (\partial_{\gamma_x}\Gamma(x,y) + \partial_{\gamma_x}\Gamma(x,y) - \partial_{\gamma_x}\Gamma(x_0,y) - \partial_{\gamma_y}\Gamma(x_0,y))\phi(y)d\sigma(y)| \\ &\leq \|\phi\|_\infty \int_{B_\delta(x_0)} |\partial_{\gamma_x}\Gamma(x,y) + \partial_{\gamma_y}\Gamma(x,y)|d\sigma(y) \\ &+ \|\phi\|_\infty \int_{B_\delta(x_0)} |\partial_{\gamma_x}\Gamma(x_0,y) + \partial_{\gamma_y}\Gamma(x_0,y)|d\sigma(y) \\ &+ \|\phi\|_\infty \int_S |\partial_{\gamma_x}\Gamma(x,y) + \partial_{\gamma_x}\Gamma(x,y) - \partial_{\gamma_x}\Gamma(x_0,y) - \partial_{\gamma_y}\Gamma(x_0,y))|d\sigma(y) \end{aligned}$$

However,

$$\partial_{\gamma_x}\Gamma(x,y) + \partial_{\gamma_y}\Gamma(x,y) = \frac{(x-y)\cdot(\gamma_{x_0} - \gamma_y)}{\omega_n|x-y|^n} \leq C|x_0 - y|^{2-n}$$

Again by S is class of C^2. Then the first and second integal is dominated by $C\delta$, the remainder is $\epsilon - \delta$ argument.
Hence we have

$$T_K\phi(x) + T_K^*\phi(x) = v_-(x) + \partial_{\gamma-}u(x) = \frac{1}{2}\phi(x) + T_K\phi(x) + \partial_{\gamma-}u(x)$$

i.e

$$\partial_{\gamma-}u = -\frac{1}{2}\phi + T_K^*\phi$$

also we have

$$T_K\phi(x) + T_K^*\phi(x) = v_+(x) + \partial_{\gamma+}u(x) = -\frac{1}{2}\phi(x) + T_K\phi(x) + \partial_{\gamma+}u(x)$$

i.e

$$\partial_{\gamma+}u = \frac{1}{2}\phi + T_K^*\phi$$

The convergence is uniform of $\partial_\gamma u(x + t\gamma)$ to $\partial_{\gamma\pm}u(x)$ in x since the same is true of v and $v + \partial_\gamma u(x)$.

□

6. Solutions of the Problems

For simplicity, we consider the operator T_K and T_K^* on $L^2(S)$ and define following subspaces

$$U_+ = ker(-\frac{1}{2}I + T_K)$$

$$U_- = ker(\frac{1}{2}I + T_K)$$

$$W_+ = ker(-\frac{1}{2}I + T_K^*)$$

$$W_- = ker(\frac{1}{2}I + T_K^*)$$

$$W_+^0 = \left\{ \beta \in W_+ : \int_S \beta = 0 \right\}$$

We define functions $\alpha_1, \cdots, \alpha_m$ and $\alpha'_1, \cdots, \alpha'_{m'}$ on S by

$$\alpha_j = \begin{cases} 1 & if\ x \in \partial\Omega_j, \\ 0 & otherwise. \end{cases}$$

$$\alpha'_j = \begin{cases} 1 & if\ x \in \partial\Omega'_j, \\ 0 & otherwise. \end{cases}$$

We first prove a key lemma which contains almost information of above subspaces.

Lemma 6.1. *The spaces U_+ and W_+ have dimension m, the spaces W_- and W_- have dimension m'. Moreover:*
(a).If $n > 2$, for each $(a_1, \cdots, a_m) \in \mathbb{C}^m$ there is a unique $\beta \in W_+$ such that the single layer potential ω with moment β satisfies $\omega|\Omega_j = a_j$ for $j = 1, \cdots, m$.
(b).If $n = 2$, there is an $(m-1)$-dimension subspace X of $\mathbb{C}^m$ such that:
i. $\mathbb{C}^m = X \oplus \mathbb{C}(1, \cdots, 1)$.
ii.for each $(a_1, \cdots, a_m) \in X$, there is a unique $\beta \in W_+^0$ such that the single layer potential ω with moment β satisfies $\omega|\Omega_j = a_j$ for $j = 1, \cdots, m$.
(c).For each $(a_1, \cdots, a_{m'}) \in \mathbb{C}^{m'}$ there is a unique $\beta \in W_-$ such that the single layer potential ω with moment β satisfies $\omega|\Omega'_j = a_j$ for $j = 1, \cdots, m'$ and $\omega|\Omega'_0 = 0$.
(d).$L^2(S) = U_+^\perp \oplus W_+ = U_-^\perp \oplus W_-$.
(e).$L^2(S) = U_+ \oplus range(-\frac{1}{2}I + T_K) = U_- \oplus range(\frac{1}{2}I + T_K)$.

Proof. a, b and c:By a simple computation, we can see $\alpha_j \in U_+$ and $\alpha'_j \in U_-$. Clearly $\alpha_1, \cdots, \alpha_m$, $\alpha'_1, \cdots, \alpha'_{m'}$ are linear independent, respectively. So $dimU_+ = dimW_+ \geq m$ and $dimU_- = dimW_- \geq m'$. On the other hand, suppose $\beta \in W_+$, let ω be the single layer potential with moment β. Hence $\partial_{\gamma^-}\omega = 0$ and ω is constant in each Ω_j, so we can define a linear map from W_+ to $\mathbb{C}^n$:

$$\beta \longrightarrow (\omega|\Omega_1, \cdots, \omega|\Omega_m)$$

If $n > 2$, it's clear that the map is injective by uniqueness of exterior Dirichlet problem which yields (a) holds.

If $n=2$, the restriciton of the map to W_+^0 is injective and its range does not contain the vector $(1,\cdots,1)$, then the (b) holds.
For W_-, the map should be from W_- to $\mathbb{C}^n$:

$$\beta \longrightarrow (\omega|\Omega'_1,\cdots,\omega|\Omega'_{m'})$$

notice $\omega=0$ in Ω'_0(by $\partial_{\gamma+}\omega$=0) and is constant in each Ω'_j (even $n=2$ by a2,b1 in lemma5.1). Hence if $\omega=0$ in each Ω'_j, then $\omega=0$ in Ω by uniqueness of interior Dirichlet problem which yields that above map is injective i.e (c) holds.
d:It only need to check $U_+^\perp\cap W_+=0$. $\forall\phi\in U_+^\perp\cap W_+$, then $T_K^*\phi=\frac{1}{2}\phi$ and $\phi=-\frac{1}{2}\psi+T_K^*\psi$, u and v are the single layer potential of ϕ and ψ respectively. Then we have

$$\partial_{\gamma-}u=0;\qquad \partial_{\gamma-}v=\phi=\frac{1}{2}\phi+T_K^*\phi=\partial_{\gamma+}u$$

Hence

$$0=\int_S u\partial_{\gamma-}v-v\partial_{\gamma-}u=\int_S u\partial_{\gamma+}u=-\int_{\Omega'}|\nabla u|^2$$

then u is locally constant in Ω' and $\phi=0$.
(Green's formula holds at last equality above by:$\int_S\phi=0$ which yields u is harmonic at infinity for $n=2$)
Then we check $U_-^\perp\cap W_-=0$. $\forall\phi\in U_-^\perp\cap W_-$, then $T_K^*\phi=-\frac{1}{2}\phi$ and $\phi=\frac{1}{2}\psi+T_K^*\phi$, u and v are the single layer potential of ϕ and ψ respectively. Then we have

$$\partial_{\gamma+}u=0;\qquad \partial_{\gamma+}v=\phi=\frac{1}{2}\phi-T_K^*\phi=-\partial_{\gamma-}u$$

Hence

$$0=\int_S u\partial_{\gamma+}v-v\partial_{\gamma+}u=\int_S -u\partial_{\gamma-}u=-\int_{\Omega}|\nabla u|^2$$

then u is locally constant in Ω and $\phi=0$.
(Green's formula holds at the first equality above by:$\int_S\phi=\int_S\psi=0$ which yields u and v is harmonic at infinity for $n=2$)
e:Just prove $L^2(S)=U_+\oplus W_+^\perp=U_-\oplus W_-^\perp$.
By:
$\forall\phi\in U_+\cap W_+^\perp$, $(\phi,\phi)=0$ by $\phi=\phi_1+\phi_2$, $\phi_1\in U_+^\perp$, $\phi_2\in W_+$. Hence $\phi=0$.
$\forall\phi\in U_-\cap W_-^\perp$, $(\phi,\phi)=0$ by $\phi=\phi_1+\phi_2$, $\phi_1\in U_-^\perp$, $\phi_2\in W_-$. Hence $\phi=0$.
Then the direct sum by linear algebra.

□

Theorem 6.2. *With the notation and terminology of section2, we have*
(I)The interior Dirichlet problem has a unique solution for every $f\in C(S)$.
(II)The exterior Dirichlet problem has a unique solution for every $f\in C(S)$.
(III)The interior Neumann problem has a solution for $f\in C(S)$ iff $\int_{\partial\Omega_j}f=0$ for $j=1,\cdots,m$. The solution is unique modulo functions which are constants on each Ω_j.
(IV)The exterior Dirichlet problem has a solution for $f\in C(S)$ iff $\int_{\partial\Omega'_j}f=0$ for $j=1,\cdots,m'$ and also for $j=0$ in case $n=2$. The solution is unique modulo

functions which are constants on each Ω'_j for $j = 1, \cdots, m'$ and also on Ω'_0 in case $n = 2$.

Proof. a:By (e) of lemma6.1, $\forall f \in C(S)$, $f = \frac{1}{2}\phi + T_K\phi + \sum_{j=1}^{m'} c_j\alpha'_j$, hence ϕ is continuous by lemma3.1(VII).
by lemma6.1(c), $\exists \beta \in W_-$, ω is the single layer potential with moment β, then

$$\omega|_S = \sum_{j=1}^{m'} c_j\alpha'_j$$

$u = v + \omega$ is the solution of interior Dirichlet problem with boundary function f, here v is the double layer potential with moment ϕ.
b.$n > 2$, as same as (a)(remove all ' and change $\frac{1}{2}$ into $-\frac{1}{2}$).
$n = 2$,

$$f = -\frac{1}{2}\phi + T_K\phi + \sum_{j=1}^{m'} c_j\alpha_j = -\frac{1}{2}\phi + T_K\phi + \sum_{j=1}^{m'} d_j\alpha_j + c\mathbb{1}_S$$

here vector $(d_1, \cdots, d_m) \in X$.
Then v is the double layer potential with moment ϕ, ω is the single potential(with moment $\beta \in W_-$) solves $\sum_{j=1}^{m'} d_j\alpha_j$ by lemma6.1(b). Finally, the solution of exterior Dirichlet problem is $v + \omega + c$.
c:just consider the sufficient condition.

$$\int_{\Omega_j} f = 0 \quad \forall j \Leftrightarrow f \in U_+^\perp = range(-\frac{1}{2}I + T_K^*) \Rightarrow -\frac{1}{2}\phi + T_K^*\phi = f$$

for some $\phi \in C(S)$.
d:just consider the sufficient condition.

$$\int_{\Omega_j} f = 0 \quad \forall j \Leftrightarrow f \in U_-^\perp = range(\frac{1}{2}I + T_K^*) \Rightarrow \frac{1}{2}\phi + T_K^*\phi = f$$

for some $\phi \in C(S)$.
$n = 2$, the single potential is harmonic at infinity implies $\int_{\partial\Omega'_0} f = 0$.

□

7. Appendix

In the appendix, we summarize some some common facts about removable singularity and asymptotic behavior at infinity of harmonic function and its radical derivative which have used before. For self-contained exposition, we give their proofs here.

Proposition 7.1. *If u is harmonic on the complement of a bounded set in $\mathbb{R}^n$, the following are equivalent:*
(I)u is harmonic at infinity.
(II)$u(x) \to 0$ as $x \to \infty$ if $n > 2$, or $|u(x)| = o(\log(|x|))$ as $x \to \infty$ if $n = 2$.
(III)$|u(x)| = O(|x|^{2-n})$ as $x \to \infty$.

Proof. we follow (I)⇒(III)⇒(II)⇒(I).
(I)⇒(III):notice that

$$\Delta u = 0 \quad in \quad \Omega \Longleftrightarrow \Delta \tilde{u} = 0 \quad in \quad \tilde{\Omega}$$

here $\tilde{\Omega}$ is some neiborhood of origin, hence (III) holds.
(III)⇒(II):trivial.
(II)⇒(I):by (II) we have

$$|\tilde{u}(x)| = \begin{cases} o(|x|^{2-n}) & if n > 2, \\ o(\log(|x|)) & if n = 2. \end{cases}$$

in some neiborhood of origin, which yields (I).

□

Proposition 7.2. *If u is harmonic at infinity, then $|\partial_r u(x)| = O(|x|^{1-n})$ as $x \to \infty$; in case $n = 2$, $|\partial_r u(x)| = O(|x|^{-2})$ as $x \to \infty$.*

Proof. By scaling transformation, we may assume that u is harmonic outside $B_{\frac{1}{2}}(0)$. Then $\tilde{u}$ is harmonic in $B_2(0)$, we can expand it in spherical harmonic function in B_1:

$$\tilde{u}(x) = \sum_{k=0}^{\infty} |x|^k Y_k(\frac{x}{|x|}) \qquad (Y_k \in H_k)$$

here H_k is the set of all spherical harmonic functions. By the relation of $\tilde{u}$ and u, we have

$$u(x) = \sum_{k=0}^{\infty} |x|^{2-n-k} Y_k(\frac{x}{|x|}) \qquad (Y_k \in H_k)$$

i.e

$$u(x) = \sum_{k=0}^{\infty} r^{2-n-k} Y_k(y) \qquad (Y_k \in H_k)$$

here $x = ry$,$r = |x|$. Hence

$$\partial_r u(x) = r^{1-n} \sum_{k=0}^{\infty} (2-n-k) r^{-k} Y_k(y) \qquad (Y_k \in H_k)$$

then for $r \geq 3$

$$|\partial_r u(x)| \leq C r^{1-n} \sum_{k=0}^{\infty} r^{-k} |Y_k(y)| \qquad (Y_k \in H_k)$$

notice that RHS converges uniformly, hence is bounded which yields our desired. For $n = 2$, the first item in sum vanishes, then we complete the proof. □

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